

General divisor function inequalities and the third cumulant

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Abstract

We extend a lower bound of Munshi on sums over divisors of a number n which are less than a fixed power of n from the squarefree case to the general case. In the process we prove a lower bound on the entropy of a geometric distribution with finite support, as well as a lower bound on the probability that a random variable is less than its mean given that it satisfies a natural condition related to its third cumulant.

1 Introduction

We consider the following problem: for which β, δ, s does the inequality

$$\tau(n)^s \ll_{\beta, \delta, s} \sum_{\substack{d|n \\ d \leq n^\delta}} \tau(d)^\beta \quad (1)$$

hold for all positive integers n ? Munshi [4] solves this problem for $s = 1$ and n squarefree: if $0 < \delta \leq \frac{1}{2}$ and

$$\beta > \frac{1 - H(\delta)}{\delta},$$

where

$$H(\delta) = \delta \log_2 \left(\frac{1}{\delta} \right) + (1 - \delta) \log_2 \left(\frac{1}{1 - \delta} \right),$$

then inequality (1) holds for squarefree n . In fact, Munshi's argument easily generalizes to any s if we require that

$$\beta > \frac{s - H(\delta)}{\delta},$$

and this is best possible by the same reasoning as in [4].

In this paper we generalize Munshi's argument to arbitrary natural numbers n . Our main result, proved in section 4, is the following.

Theorem 3. *If $0 < \delta \leq \frac{1}{2}$, $\beta, s \geq 0$ satisfy*

$$\beta > \frac{s - H(\delta)}{\delta},$$

then

$$\tau(n)^s \ll_{\beta, \delta, s} \sum_{\substack{d|n \\ d \leq n^\delta}} \tau(d)^\beta.$$

The main new idea in the proof is to sample divisors d of n from a probability distribution having high entropy, while keeping the average value of $\log(d)$ small. A crucial ingredient in the proof is the following entropy inequality, which is proved in section 3.

Corollary 3. *If X is geometrically distributed on $\{0, \dots, m\}$ with mean δm then*

$$H(X) \geq \log_2(m+1)H(\delta),$$

and the inequality is strict if $m > 1$ and $\delta \notin \{0, \frac{1}{2}, 1\}$.

In the process of proving our main result, we also prove a variation on a related inequality due to Soundararajan. Suppose that δ is a real number between 0 and 1, and define $c(\delta)$ to be the largest real number such that, for any squarefree number n , we have the inequality

$$\sum_{\substack{d|n \\ d \leq n^\delta}} \delta^{\omega(d)} (1-\delta)^{\omega(n/d)} \geq c(\delta). \quad (2)$$

Taking n to have k prime factors that are sufficiently close in size, we see that

$$\delta < \frac{1}{k} \implies c(\delta) \leq (1-\delta)^k.$$

Soundararajan has shown in [6] (with different notation - his $A(t)$ is our $c(1/(1+t))$, and his $B(t)$ is our $1 - c(t/(1+t))$) the following recursive inequalities:

$$\begin{aligned} c\left(\frac{\delta}{1+\delta}\right) &\geq \frac{c(\delta)}{1+c(\delta)}, \\ c\left(\frac{1}{2-\delta}\right) &\geq \frac{c(\delta)}{1+c(\delta)}. \end{aligned}$$

Using these together with the obvious bound $c(1) = 1$, he shows that $c(1 - 1/k) = 1/k$ for $k \in \mathbb{N}$, and that if δ is rational with continued fraction $[a_0, a_1, \dots, a_r]$ then

$$c(\delta) \geq \frac{1}{a_0 + \dots + a_r}.$$

Definition 1. For any integers $n \geq k$, define $g(n, k)$ by

$$g(n, k) = \min_{a_1 + \dots + a_n = 0} |\{S \subseteq \{1, \dots, n\} \mid |S| = k, \sum_{i \in S} a_i \geq 0\}|.$$

Conjecture (Manickam-Miklós-Singhi). *For $n \geq 4k$, $g(n, k) = \binom{n-1}{k-1}$.*

The Manickam-Miklós-Singhi conjecture has been proved in the cases $k \leq 7$ [3], $n \geq 2k^3$, $n \geq 33k^2$ [1], and $n \geq 10^{46}k$ [5]. In [3], a slightly stronger conjecture is made based on numerical evidence: if $\binom{n-1}{k-1} \leq \binom{n-3}{k}$, then $g(n, k) = \binom{n-1}{k-1}$. The MMS conjecture is related to $c(\delta)$ by the following easy proposition.

Proposition 1. *If $\delta = \frac{a}{b}$, then*

$$c(\delta) \geq \frac{g(b, a)}{\binom{b}{a}}.$$

In particular, if $g(b, a) = \binom{b-1}{a-1}$ then $c(\delta) \geq \delta$. If the stronger version of the MMS conjecture proposed in [3] holds, then $c(\delta) \geq \delta$ for all $\delta \leq (1 - \delta)^3$.

In the next section, we prove that

$$\delta \leq \frac{1}{2} \implies c(\delta) \geq \frac{1}{2e^{3/2}}.$$

In fact, we prove the following stronger claim.

Corollary 1. *Let X_1, \dots, X_n be independent random variables supported on \mathbb{N} such that for each i the function $k \mapsto \mathbb{P}[X_i = k]$ is decreasing, and let $w_1, \dots, w_n \geq 0$. Let $X = \sum_{i=1}^n w_i X_i$. Then $\mathbb{P}[X \leq \mathbb{E}[X]] \geq \frac{1}{2e^{3/2}}$.*

2 Lower bound on the probability that a random variable is less than its mean

The arguments in this section are inspired by a MathOverflow post of fedja [2], which used Bernstein's trick in a similar way to solve a closely related problem.

Consider the following property which a random variable X might have:

$$\forall t \geq 0 \quad \frac{d^3}{dt^3} \log (\mathbb{E}[e^{-tX}]) \leq 0. \tag{P}$$

If independent random variables X_1, \dots, X_n all have property (P), and if $w_1, \dots, w_n \geq 0$, then the random variable $X = \sum_i w_i X_i$ also has property (P) by Bernstein's trick.

Note that when $t = 0$, property (P) says that the third cumulant of X , $\mathbb{E}[(X - \mathbb{E}[X])^3]$, is at least zero.

Theorem 1. *If a random variable X has property (P), then*

$$\mathbb{P}[X \leq \mathbb{E}[X]] \geq \frac{1}{2e^{3/2}}.$$

Proof. Let $Y = \mathbb{E}[X] - X$, and define the function $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ by

$$g(t) = \log (\mathbb{E}[e^{tY}]).$$

Property (P) says that $g''(t)$ is decreasing for $t \geq 0$. Since $\mathbb{E}[Y] = 0$, we also have $g'(0) = 0$, and so for any $t \geq 0$ we have

$$tg''(t) \leq \int_0^t g''(x)dx = g'(t).$$

Integrating this we see that $tg'(t) \leq 2g(t)$. The inequality $tg''(t) \leq g'(t)$ is easily seen to be equivalent to

$$\frac{\mathbb{E}[(tY)^2 e^{tY}]}{\mathbb{E}[tY e^{tY}]} \leq \frac{\mathbb{E}[tY e^{tY}]}{\mathbb{E}[e^{tY}]} + 1.$$

If g is not identically 0 then we can find t such that $tg'(t) = 1$, or equivalently such that $\mathbb{E}[tY e^{tY}] = \mathbb{E}[e^{tY}] = e^{g(t)}$. For this t we have

$$\mathbb{E}[tY(8 - 3tY)e^{tY}] \geq (8 - 6)e^{g(t)} \geq 2e^{1/2}.$$

The function $p(x) = x(8 - 3x)e^x$ has $p(x) \leq 0$ for $x \leq 0$ and $p(x) \leq p(2) = 4e^2$ for all x , so by Markov's inequality

$$\mathbb{P}[X \leq \mathbb{E}[X]] = \mathbb{P}[Y \geq 0] \geq \frac{1}{2e^{3/2}}. \quad \square$$

Theorem 2. *Suppose that the random variable X is supported on \mathbb{N} , with $\mathbb{P}[X = k]$ a decreasing function of k . Then X satisfies property (P).*

Proof. Expanding property (P), it becomes

$$\mathbb{E}[X^3 e^{-tX}] \mathbb{E}[e^{-tX}]^2 + 2\mathbb{E}[X e^{-tX}]^3 \geq 3\mathbb{E}[X^2 e^{-tX}] \mathbb{E}[X e^{-tX}] \mathbb{E}[e^{-tX}].$$

Setting $a_k = \mathbb{P}[X = k]$ and $x = e^{-t}$ we get the polynomial inequality

$$\sum_{i,j,k} a_i a_j a_k x^{i+j+k} (i^3 + 2ijk - 3i^2 j) \geq 0,$$

which we need to check for $a_0 \geq a_1 \geq \dots \geq 0$ and $1 \geq x \geq 0$. The left hand side of the above is equal to

$$\sum_{i < j} a_i a_j x^{i+j} (a_i x^i - a_j x^j) (j-i)^3 + \sum_{i < j < \frac{i+k}{2}} a_i a_k x^{i+k} (a_j x^j - a_{i+k-j} x^{i+k-j}) (i+k-2j) (j+k-2i) (2k-i-j),$$

which is obviously nonnegative. \square

Corollary 1. *Let X_1, \dots, X_n be independent random variables supported on \mathbb{N} such that for each i the function $k \mapsto \mathbb{P}[X_i = k]$ is decreasing, and let $w_1, \dots, w_n \geq 0$. Let $X = \sum_{i=1}^n w_i X_i$. Then $\mathbb{P}[X \leq \mathbb{E}[X]] \geq \frac{1}{2e^{3/2}}$.*

Corollary 2. *Let $\delta \leq \frac{1}{2}$, and let $f : \mathbb{N} \rightarrow [0, \infty)$ be a nonnegative multiplicative function such that for every prime p we have $\frac{f(p)}{f(p)+1} \leq \delta$. Then for any squarefree number n we have*

$$\sum_{\substack{d|n \\ d \leq n^\delta}} f(d) \geq \frac{1}{2e^{3/2}} \sum_{d|n} f(d).$$

3 A lower bound for the entropies of certain probability distributions

Let X be a random variable supported the set $\{0, \dots, m\}$, with probability distribution $\rho = (\rho_0, \dots, \rho_m)$. We define the entropy of X to be

$$H(X) = \sum_{i=0}^m \rho_i \log_2 \left(\frac{1}{\rho_i} \right).$$

In the next section, we will need the existence of a random variable X as above with $\mathbb{E}[X] = \delta m$ given and $H(X)$ large. It's a well-known fact that the optimal choice of X will be geometrically distributed. Unfortunately the entropy of a geometric distribution on a finite set, as a function of the mean, is quite complicated and directly proving a lower bound is rather difficult. Instead, we will inductively construct probability distributions which are simpler to analyze and still have sufficiently large entropy.

Lemma 1. *For every $m \geq 1$ and every $0 \leq \delta \leq 1$ there is a random variable X supported on the set $\{0, \dots, m\}$ which has mean δm and entropy satisfying the inequality*

$$H(X) \geq \log_2(m+1)H(\delta).$$

Proof. It's enough to prove this for $0 < \delta \leq \frac{1}{2}$. We proceed by induction on m . The case $m = 1$ is immediate. For $m > 1$, we let Y be a random variable on the set $\{0, \dots, m-1\}$ with mean $\delta(m-1)$, satisfying $H(Y) \geq \log_2(m)H(\delta)$. Define X to be 0 with probability $\frac{1-\delta}{m\delta+1-\delta}$, and to be $1+Y$ with probability $\frac{m\delta}{m\delta+1-\delta}$. Then

$$\mathbb{E}[X] = \frac{m\delta}{m\delta+1-\delta} (1 + \mathbb{E}[Y]) = \frac{m\delta}{m\delta+1-\delta} (1 + \delta(m-1)) = \delta m,$$

and

$$H(X) = H\left(\frac{1-\delta}{m\delta+1-\delta}\right) + \frac{m\delta}{m\delta+1-\delta} H(Y) \geq H\left(\frac{1-\delta}{m\delta+1-\delta}\right) + \frac{m\delta}{m\delta+1-\delta} \log_2(m)H(\delta).$$

It suffices to show that the right hand side of the above is at least $\log_2(m+1)H(\delta)$.

Making the change of variables $x = \frac{1-\delta}{m\delta}$, we just need to show

$$H\left(\frac{1}{x+1}\right) \geq \left(\log_2(m+1) - \frac{\log_2(m)}{x+1}\right) H\left(\frac{1}{mx+1}\right)$$

for real numbers m, x satisfying $m \geq 1$ and $mx \geq 1$. Since we clearly have equality when $m = 1$ or $m = \frac{1}{x}$, it is enough to show that the right hand side is a decreasing function of m . Taking the derivative with respect to m , using the identity $H'(\delta) = \log_2\left(\frac{1-\delta}{\delta}\right)$, we see that we just need to check

$$\left(\log_2(m+1) - \frac{\log_2(m)}{x+1}\right) \frac{x \log_2(mx)}{(mx+1)^2} \geq \log_2(e) \left(\frac{1}{m+1} - \frac{1}{m+mx}\right) H\left(\frac{1}{mx+1}\right).$$

Changing variables back to m, δ and rearranging, this becomes

$$(1-\delta) \left(1 + \frac{1}{m}\right) \log_2(m+1) + \delta(m+1) \log_2\left(1 + \frac{1}{m}\right) \geq \frac{\log_2(e)(1-2\delta)H(\delta)}{\delta(1-\delta) \log_2\left(\frac{1-\delta}{\delta}\right)}.$$

From $(1-\delta) \geq \delta$ and $m \geq 1$ we easily deduce that the left hand side is at least 2, so it is enough to prove the single variable inequality

$$2\delta(1-\delta) \frac{\log(1-\delta) - \log(\delta)}{(1-\delta) - \delta} \geq H(\delta),$$

where the logarithms on the left hand side are taken to the base e . We leave this inequality as an exercise for the reader. \square

Corollary 3. *If X is geometrically distributed on $\{0, \dots, m\}$ with mean δm then*

$$H(X) \geq \log_2(m+1)H(\delta),$$

and the inequality is strict if $m > 1$ and $\delta \notin \{0, \frac{1}{2}, 1\}$.

Proof. This follows from the previous lemma together with the well-known fact that a geometric distribution has the maximum entropy among all distributions on a finite set which have a given mean. \square

Remark 1. In the case $m+1 = 2^k$ we can give a much simpler proof of Lemma 1. Let B_0, \dots, B_{k-1} be i.i.d. random variables which are each 0 with probability $1 - \delta$ and 1 with probability δ . Then if we take

$$X = \sum_{i=0}^{k-1} 2^i B_i,$$

we have $\mathbb{E}[X] = \delta m$ and $H(X) = kH(\delta)$. This probability distribution corresponds to a trick used by Wolke in [8].

4 Divisor sum inequalities

Theorem 3. *If $0 < \delta \leq \frac{1}{2}$, $\beta, s \geq 0$ satisfy*

$$\beta > \frac{s - H(\delta)}{\delta},$$

then

$$\tau(n)^s \ll_{\beta, \delta, s} \sum_{\substack{d|n \\ d \leq n^\delta}} \tau(d)^\beta.$$

Proof. Choose a number M such that for all $m \geq M$ we have

$$\beta > \frac{s - \frac{\lfloor \log_2(m+1) \rfloor}{\log_2(m+1)} H(\delta)}{\delta}.$$

Write $n = \prod_i p_i^{m_i}$. We define a collection of independent random variables X_i , X_i taking values in $\{0, \dots, m_i\}$, as follows. If $m_i < M$, we take X_i to be geometrically distributed with mean δm_i . If $m_i \geq M$, choose k such that $2^k - 1 \leq m_i < 2^{k+1} - 1$, and let B_0, \dots, B_{k-1} be k i.i.d. random variables which are each 0 with probability $1 - \delta$ and 1 with probability δ . Set

$$X_i = \left(\sum_{j=0}^{k-2} 2^j B_j \right) + (m_i + 1 - 2^{k-1}) B_{k-1}.$$

Finally, we define a random variable D dividing n by $D = \prod_i p_i^{X_i}$.

We have

$$\mathbb{E}[\log(D)] = \sum_i \mathbb{E}[X_i] \log(p_i) = \delta \log(n),$$

so by Corollary 1 we have

$$\mathbb{P}[D \leq n^\delta] \geq \frac{1}{2e^{3/2}}.$$

Setting $P_n(d) = \mathbb{P}[D = d]$, this can be written as

$$1 \leq 2e^{3/2} \sum_{\substack{d|n \\ d \leq n^\delta}} P_n(d).$$

By Hölder's inequality, for any $t > 0$ we have

$$\sum_{\substack{d|n \\ d \leq n^\delta}} P_n(d) \leq \left(\sum_{d|n, d \leq n^\delta} \tau(d)^\beta \right)^{\frac{1}{t+1}} \left(\sum_{d|n} P_n(d)^{\frac{t+1}{t}} \tau(d)^{-\frac{\beta}{t}} \right)^{\frac{t}{t+1}}.$$

Combining the last two inequalities, we see that

$$\left(\sum_{d|n} P_n(d)^{\frac{t+1}{t}} \tau(d)^{-\frac{\beta}{t}} \right)^{-t} \leq (2e^{3/2})^{t+1} \sum_{\substack{d|n \\ d \leq n^\delta}} \tau(d)^\beta.$$

To finish, we just need to choose t large enough that the left hand side of the above is at least $\tau(n)^s$. Since the left hand side is a multiplicative function of n , we can restrict to the case $n = p^m$, with just a single probability distribution X on the possible exponents $\{0, \dots, m\}$. Write $\rho_m(x)$ for $P_{p^m}(p^x) = \mathbb{P}[X = x]$. Then we just need to choose t large enough to make the inequality

$$(m+1)^s \leq \left(\sum_{x=0}^m \rho_m(x)^{\frac{t+1}{t}} (x+1)^{-\frac{\beta}{t}} \right)^{-t} \quad (3)$$

hold for all $m \geq 1$. We have

$$\lim_{t \rightarrow \infty} \left(\sum_{x=0}^m \rho(x) \left(\frac{(x+1)^\beta}{\rho(x)} \right)^{-\frac{1}{t}} \right)^{-t} = \prod_{x=0}^m \left(\frac{(x+1)^\beta}{\rho(x)} \right)^{\rho(x)} = 2^{H(X) + \beta \mathbb{E}[\log_2(X+1)]}.$$

Since \log_2 is a concave function, we have

$$\mathbb{E}[\log_2(X+1)] \geq \mathbb{E} \left[\frac{X}{m} \log_2(m+1) + \left(1 - \frac{X}{m} \right) \log_2(1) \right] = \delta \log_2(m+1).$$

Thus, by the assumption on β and Corollary 3 we can find a t_0 such that for any $t \geq t_0$ and any $m < M$ inequality (3) is satisfied. For $m \geq M$, we use the easy inequality

$$\left(\sum_{x=0}^m \rho(x)^{\frac{t+1}{t}} (x+1)^{-\frac{\beta}{t}} \right)^{-t} \geq \left((1-\delta)^{\frac{t+1}{t}} + \delta^{\frac{t+1}{t}} 2^{-\frac{\beta}{t}} \right)^{-t \lfloor \log_2(m+1) \rfloor},$$

which follows from the fact that for any x , $x+1$ is at least 2^B , where B is the number of 1s in the binary representation of x . Thus if we take t large enough to make

$$\left((1-\delta)^{\frac{t+1}{t}} + \delta^{\frac{t+1}{t}} 2^{-\frac{\beta}{t}} \right)^{-t}$$

sufficiently close to $2^{H(\delta) + \beta \delta}$, then inequality (3) will be satisfied for $m \geq M$ as well. \square

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References

- [1] Noga Alon, Hao Huang, and Benny Sudakov. Nonnegative k -sums, fractional covers, and probability of small deviations. *J. Combin. Theory Ser. B*, 102(3):784–796, 2012.
- [2] fedja (<http://mathoverflow.net/users/1131/fedja>). lower-bound for $Pr[X \geq EX]$. MathOverflow. URL:<http://mathoverflow.net/q/188087> (version: 2014-11-26).
- [3] Stephen G. Hartke and Derrick Stolee. A linear programming approach to the Manickam–Miklós–Singhi conjecture. *European J. Combin.*, 36:53–70, 2014.
- [4] Ritabrata Munshi. Inequalities for divisor functions. *Ramanujan J.*, 25(2):195–201, 2011.
- [5] A. Pokrovskiy. A linear bound on the Manickam-Miklos-Singhi Conjecture. *ArXiv e-prints*, August 2013.
- [6] K. Soundararajan. An inequality for multiplicative functions. *J. Number Theory*, 41(2):225–230, 1992.
- [7] J. G. van der Corput. Une inégalité relative au nombre des diviseurs. *Nederl. Akad. Wetensch., Proc.*, 42:547–553, 1939.
- [8] Dieter Wolke. A new proof of a theorem of van der Corput. *Journal of the London Mathematical Society*, s2-5(4):609–612, 1972.